Are excitons really bosons?

B Laikhtman

Racah Institute of Physics, Hebrew University, Jerusalem 91904, Israel

Abstract
Non-bosonic commutation relations of exciton operators discovered by Keldysh and Kozlov (1968 *Zh. Eksp. Teor. Fiz.* **54** 978; 1968 *Sov. Phys.—JETP* **27** 521 (Engl. Transl.)) led to a long-standing question about the possible difference between excitons and bosons in spite of the integer spin of excitons. In this paper the problem is analysed and it is shown that, in spite of the difference between separate excitons and bosons, the exciton gas is a Bose gas. Bosons comprising this gas are mixtures of separate excitons, and the non-bosonic nature of excitons leads only to a renormalization of the interaction between them. Features of the exciton luminescence coming from the interaction and non-bosonic nature of excitons are considered.

1. Introduction

The fact that an exciton as a bound state of two Fermi particles has an integer spin immediately suggests that the properties of an exciton gas are similar to the properties a Bose gas. This was first noticed by Moskalenko, who suggested that excitons can form a Bose condensate [1].

However, the situation appears not that simple when one tries to describe an exciton gas with the help of a technique developed for a Bose gas. Keldysh and Kozlov [2] (see also [3–9]) introduced creation and annihilation exciton operators as

$$c^{\dagger}_{ps\nu} = \frac{1}{\sqrt{V}} \sum_{k} \phi^{*}_{k\nu} b^{\dagger}_{(m_h/M)p-k,\sigma_h} a^{\dagger}_{(m_e/M)p+k,\sigma_e}, \qquad (1.1a)$$

$$c_{ps\nu} = \frac{1}{\sqrt{V}} \sum_{k} \phi_{k\nu} a_{(m_e/M)p+k,\sigma_e} b_{(m_h/M)p-k,\sigma_h}, \qquad (1.1b)$$

where a_{k,σ_e} and b_{k,σ_h} are the electron and hole annihilation operators, σ_e and σ_h are electron and hole spin quantum numbers, $s = \sigma_e + \sigma_s$ is the exciton spin quantum number, m_e and m_h are the electron and hole masses, $M = m_e + m_h$ is the exciton mass, p is the exciton momentum, $\phi_{\nu}(k)$ is the Fourier component of the exciton wavefunction,

$$\phi_{k\nu} = \int e^{-ikr} \phi_{\nu}(r) \, dr, \qquad (1.2a)$$

$$-\frac{\hbar^2}{2m} \phi_{\nu}(r) + U(r)\phi_{\nu}(r) = E_{\nu}\phi_{\nu}(r), \qquad (1.2b)$$

$m = m_e m_h / M$ is the reduced electron–hole mass, $U(r)$ is the interaction potential between electrons and holes, and ν is an internal quantum number of the exciton state. Operating on the vacuum wavefunction operator $c_{p\nu}^\dagger$ creates the exciton wavefunction with momentum p and in the state ν. But, in spite of the integer spin of excitons, the commutation relations of the exciton operators,

$$[c_{p\nu}, c_{p'\nu'}^\dagger] = \delta_{p,p'}\delta_{\nu,\nu'} - \frac{1}{V}\sum_k \phi_{k+(m_h/M)p,\nu}\phi_{k+(m_h/M)p',\nu'}^* a_{p'+k}^\dagger a_{p+k}$$

$$- \frac{1}{V}\sum_k \phi_{k-(m_e/M)p,\nu}\phi_{k-(m_e/M)p',\nu'}^* b_{p'-k}^\dagger b_{p-k}, \tag{1.3}$$

are different for the commutation relations for Bose operators [2, 4, 7]. Generally speaking, such a result could be expected. Indeed, according to equation (1.1), an exciton state contains contributions from all free electron and hole states. Because of this, the second electron–hole pair inevitably overlaps with the first one and creation of the second exciton is not the same as creation of the first one.

In spite of this difference between excitons and bosons, theory predicts a behaviour similar to Bose condensation [1, 2], and there is a number of claims that the Bose condensation of excitons has been detected experimentally. A natural question is why physical properties of the exciton gas are so similar to properties of a Bose gas, and if there is any difference at all between a Bose gas and an exciton gas. Is it possible, in general, that a gas of particles with non-Bose statistics behaves as a Bose gas? The absence of an answer to these questions motivated the suggestion of a new technique to deal with a gas of particles close to bosons but different from them [10].

It is necessary to note that the problem of the description of composite bosons has been studied for atoms where a few formal approaches based on canonical transformations were suggested [11–17].

The purpose of the present paper is to show that the exciton gas is physically identical to the Bose gas. Bosons that constitute the gas are not separate excitons but mixtures of different excitonic states. The difference between them and excitons makes the interaction between them different from interaction between excitons and, what is more important, modifies the exciton–electromagnetic field interaction that can be detected experimentally.

The development of the theory does not depend on the space dimensionality and, to be specific, the whole consideration below is made for the two-dimensional (2D) case just because the 2D results are simpler and most of the experiments where the statistics are important are being made currently in quantum wells.

2. Analysis of the problem

First of all, it is necessary to note that the difference between the commutation relations of the exciton operators (1.1) and the Bose operators is proportional to the density of the particles, i.e. it is small in the dilute gas and reveals itself along with interaction between excitons [2]. Therefore the question about the similarity and difference between excitons and bosons makes sense only for a non-ideal gas, while in an ideal gas the excitons are *completely* equivalent to bosons.

As a whole, the situation looks quite contradictory: a single exciton looks like a boson (integer spin!), while many excitons do not (commutation relations!). To understand the physical reason behind this formal property, let us consider the wavefunctions of exciton pairs.

A wavefunction of a single exciton,

$$\psi_\gamma(e,h) = \frac{1}{\sqrt{S}} e^{\mathrm{i} pR} g_s(\sigma_e, \sigma_h) \phi_v(\boldsymbol{r}_e - \boldsymbol{r}_h), \qquad (2.1)$$

depends on the coordinates of a single electron, $\boldsymbol{r}_e$, and a single hole, $\boldsymbol{r}_h$. Here $\boldsymbol{R} = (m_e \boldsymbol{r}_e + m_h \boldsymbol{r}_h)/M$ is the exciton centre-of-mass coordinate, g_s is the spin wavefunction, γ is the composite exciton quantum number that is the set of quantum numbers $\gamma = (psv)$, and S is the normalization area. Typically, a wavefunction of two non-interacting particles is constructed as a properly symmetrized product of two single particle wavefunctions. For excitons, this procedure leads to

$$\Phi_{\gamma_1\gamma_2}(e_1, h_1; e_2, h_2) = \frac{1}{2\sqrt{1+\delta_{\gamma_1,\gamma_2}}} [\psi_{\gamma_1}(e_1, h_1)\psi_{\gamma_2}(e_2, h_2) + \psi_{\gamma_1}(e_2, h_2)\psi_{\gamma_2}(e_1, h_1)$$
$$- \psi_{\gamma_1}(e_1, h_2)\psi_{\gamma_2}(e_2, h_1) - \psi_{\gamma_1}(e_2, h_1)\psi_{\gamma_2}(e_1, h_2)] \qquad (2.2)$$

where e_j and h_j are sets of electron and hole coordinates, respectively. The prefactor here is chosen in such a way that, if terms in the square brackets were mutually orthogonal, the function would be normalized to unity. In the space of all states of two electrons and two holes, the functions $\Phi_{\gamma_1\gamma_2}$ make up a complete basis and functions ψ_γ make up a complete basis in the space of all states of a single electron–hole pair (see the appendix). But functions $\Phi_{\gamma_1\gamma_2}$ are not orthogonal, in spite of the orthogonality of ψ_γ [18]. For example, function

$$\Phi_{\gamma_1\gamma_3}(e_1, h_1; e_2, h_2) = \frac{1}{2\sqrt{1+\delta_{\gamma_1,\gamma_3}}} [\psi_{\gamma_1}(e_1, h_1)\psi_{\gamma_3}(e_2, h_2) + \psi_{\gamma_1}(e_2, h_2)\psi_{\gamma_3}(e_1, h_1)$$
$$- \psi_{\gamma_1}(e_1, h_2)\psi_{\gamma_3}(e_2, h_1) - \psi_{\gamma_1}(e_2, h_1)\psi_{\gamma_3}(e_1, h_2)] \qquad (2.3)$$

where $\gamma_3 \neq \gamma_2$ is not orthogonal to $\Phi_{\gamma_1\gamma_2}$, because the first pair of terms of $\Phi_{\gamma_1\gamma_2}$ are not orthogonal to the second pair of terms of $\Phi_{\gamma_1\gamma_3}$ and the second pair of terms of $\Phi_{\gamma_1\gamma_2}$ are not orthogonal to the first pair of terms of $\Phi_{\gamma_1\gamma_3}$.

A similar situation also exists for $N > 2$ electron–hole pairs. It is possible to introduce a complete basis of exciton states,

$$\Phi_{\gamma_1,\ldots,\gamma_N}(e_1, h_1; \ldots; e_N, h_N) = \frac{\sqrt{N_1! N_2! \ldots}}{N!} \sum (-1)^P \prod_{j=1}^{N} \psi_{\gamma_j}(e_{j_1}, h_{j_2}) \qquad (2.4)$$

where the summation is carried out over all transpositions of j_1 and j_2 that produce non-equivalent wavefunction products, P is the parity of the transpositions, N_1, N_2 and so on are numbers equal to exciton quantum numbers, $\sum_k N_k = N$, and the prefactor equals the inverse square root of the number of terms in the sum[1]. Functions (2.4) with different quantum numbers are not orthogonal [18].

Non-orthogonality of the functions makes it impossible use them as a basis for second quantization. To make this clear, it is helpful to review the regular introduction of second quantization given in textbooks (see, e.g., [19]). The basis of the second quantization for bosons is usually constructed of symmetrized products of single boson wavefunctions $\tilde{\phi}_\alpha(\boldsymbol{r})$. If among N particles there are groups with N_1, N_2, ... respectively equal quantum numbers and $\sum N_k = N$, then the basis functions are

$$\tilde{\Phi}_{\alpha_1\ldots\alpha_N}(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) = \sqrt{\frac{N_1! N_2! \ldots}{N!}} \sum \tilde{\phi}_{\alpha_1}(\boldsymbol{r}_{i_1}) \ldots \tilde{\phi}_{\alpha_1}(\boldsymbol{r}_{i_{N_1}}) \tilde{\phi}_{\alpha_2}(\boldsymbol{r}_{i_{N_1+1}}) \ldots \tilde{\phi}_{\alpha_2}(\boldsymbol{r}_{i_{N_2}}) \ldots$$
$$(2.5)$$

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[1] There are $N!/N_1! N_2! \ldots$ non-equivalent ways to place N electrons in the product of N_1 wavefunction with quantum numbers γ_1, N_2 wavefunction with quantum number γ_2, and so on. After that, among $N!$ different placing of N holes in the product of $\sum_j N_j = N$ wavefunctions no two are equivalent.

where the summation is carried out over all transpositions of i_j or over all transpositions of α_j that produce non-equivalent function products. If $\tilde{\phi}_\alpha(r)$ are orthogonal, then $\tilde{\Phi}_{\alpha_1 \ldots \alpha_N}$ with different sets of quantum numbers are also orthogonal. These wavefunctions are completely characterized by occupation numbers of different single particle states, $N_1, N_2, \ldots$. As a result, the matrix elements of any symmetric single particle operator,

$$F = \sum_{j=1}^{N} f_j, \tag{2.6}$$

where f_j operates only on functions of the jth particle, have the form

$$\langle N_1 \ldots N_j - 1 \ldots N_k + 1 \ldots |F| N_1 \ldots N_j \ldots N_k \ldots \rangle = \langle \alpha_k | f | \alpha_j \rangle \sqrt{N_j (N_k + 1)}. \tag{2.7}$$

Factor $\sqrt{N_j (N_k + 1)}$ comes from factorial prefactors and the number of terms in the sums (2.5). This factor makes it possible to introduce creation and annihilation operators, $c_\alpha^\dagger$ and c_α, and to write operator (2.6) as

$$F = \sum_{jk} \langle \alpha_k | f | \alpha_j \rangle c_{\alpha_k}^\dagger c_{\alpha_j}. \tag{2.8}$$

This rule is generalized to many particle symmetric operators. If functions $\tilde{\phi}_\alpha(r)$ and, along with them, functions (2.5) are not orthogonal, then matrix elements of F are not reduced to form (2.7) and it is impossible to introduce creation and annihilation operators that reduce operator F to the form of (2.8).

It makes sense to point out another disadvantage of the basis (2.5) non-orthogonality. Transformation of the Schrödinger equation for an electron–hole gas,

$$\mathcal{H}\Psi = E\Psi, \tag{2.9}$$

to this basis with the help of expansion

$$\Psi = \sum_\Gamma C_\Gamma \Phi_\Gamma \tag{2.10}$$

where $\Gamma = (\gamma_1, \gamma_2, \ldots)$ results in

$$\sum_{\Gamma'} \mathcal{H}_{\Gamma\Gamma'} C_{\Gamma'} = E \sum_{\Gamma'} \mathcal{N}_{\Gamma\Gamma'} C_{\Gamma'}. \tag{2.11}$$

Here

$$\mathcal{H}_{\Gamma\Gamma'} = \langle \Phi_\Gamma | \mathcal{H} | \Phi_{\Gamma'} \rangle, \qquad \mathcal{N}_{\Gamma\Gamma'} = \langle \Phi_\Gamma | \Phi_{\Gamma'} \rangle. \tag{2.12}$$

The normalization matrix $\mathcal{N}$ is not the unity matrix, and because of this equation (2.11) does not have the form of the Schrödinger equation. This means that $\mathcal{H}_{\Gamma\Gamma'}$ cannot be considered as matrix elements of the Hamiltonian.

3. Bose operators for exciton gas

The non-orthogonality of the basis (2.2) for two excitons and basis (2.4) looks a bit strange, because the Hamiltonian of N electron–hole pairs,

$$\mathcal{H} = \sum_j \left(-\frac{\hbar^2 \nabla_{ej}^2}{2m_e} - \frac{\hbar^2 \nabla_{hj}^2}{2m_h} \right)$$

$$+ \sum_{ij} u_{eh}(|r_{ei} - r_{hj}|) + \frac{1}{2} \sum_{ij} \left[u_{ee}(|r_{ei} - r_{ej}|) + u_{hh}(|r_{hi} - r_{hj}|) \right] \tag{3.1}$$

(where r_e and r_h are electron and hole coordinates, m_e and m_h are electron and hole masses, and u_{ij} is the interaction energy between particles of the ith and jth kind) is Hermitian and its eigenfunctions are orthogonal. This means that the difficulty is not in the essence of the problem but in the poor approximation of the eigenfunctions by equations (2.2) and (2.4).

The 'non-orthogonality catastrophe' can be removed by orthogonalization of basis (2.4):

$$\Phi_\Gamma^{(\mathrm{ort})} = \sum_{\Gamma_1} \Phi_{\Gamma_1} \mathcal{N}_{\Gamma_1,\Gamma}^{-1/2}, \qquad \langle \Phi_\Gamma^{(\mathrm{ort})} | \Phi_{\Gamma'}^{(\mathrm{ort})} \rangle = \delta_{\Gamma,\Gamma'}. \tag{3.2}$$

Then the expansion

$$\Psi = \sum_\Gamma F_\Gamma \Phi_\Gamma^{(\mathrm{ort})}, \tag{3.3}$$

leads to the Schrödinger equation

$$\mathcal{H}_X F = E F \tag{3.4a}$$

with the Hamiltonian

$$\mathcal{H}_X = \mathcal{N}^{-1/2} \langle \Phi | \mathcal{H} | \Phi \rangle \mathcal{N}^{-1/2}. \tag{3.4b}$$

However, technically the exact orthogonalization of the basis and the calculation of $\mathcal{H}_X$ is not tractable. Behind the mathematical difficulty there is a physical reason. The exactly orthogonolized exciton basis describes all states of the electron–hole system, including free carrier states. The description of free states with the help of the exciton basis is even more difficult than the description of exciton states with the free carrier basis. The situation is simplified under the conditions that guarantee that most of the free carriers are bound in excitons and the unbound states can be neglected. There are two such conditions [20].

The first condition is that the average exciton kinetic energy E_k (in equilibrium this is the temperature of the exciton gas) is small compared to the exciton binding energy, E_b:

$$E_k/E_b \ll 1. \tag{3.5}$$

If this inequality is violated, most of the excitons are ionized and we deal with the electron–hole plasma.

The second condition is that the exciton radius a is small compared to the average distance between the excitons, $n^{-1/2}$, where n is the exciton concentration,

$$na^2 \ll 1. \tag{3.6}$$

When the average distance between excitons becomes comparable with their radius, i.e., they overlap, they are destroyed due to the Mott transition and the exciton gas becomes the electron–hole plasma.

If condition (3.5) is met, then excited exciton states can be discarded, i.e. the exciton internal quantum number ν corresponds to the ground state. However, the absolute neglect of the excited states makes the basis (2.5) not complete, and more accurate consideration is necessary.

To reduce the consideration to the ground-state excitons only, it is convenient to break the complete exciton basis (2.4) into two parts: (1) states where there are only ground-state single exciton wavefunctions and (2) states containing at least one wavefunction of an excited exciton state. Then equation (2.11) can be written in the following matrix form:

$$(\mathcal{H}_{11} - E\mathcal{N}_{11})\, C_1 + (\mathcal{H}_{12} - E\mathcal{N}_{12})\, C_2 = 0, \tag{3.7a}$$

$$(\mathcal{H}_{21} - E\mathcal{N}_{21})\, C_1 + (\mathcal{H}_{22} - E\mathcal{N}_{22})\, C_2 = 0, \tag{3.7b}$$

where C_1 and C_2 are states belonging to the first and the second parts of the basis, respectively. C_2 can be eliminated from equation (3.7a) with the help of equation (3.7b), leading to an equation that formally contains only ground-state exciton wavefunctions:

$$(\mathcal{H}_{11} - E\mathcal{N}_{11})\, C_1 - \mathcal{H}^{(XX)} C_1 = 0. \tag{3.8}$$

The last term in this equation describes the effect of the excited states,

$$\mathcal{H}^{(XX)} = (\mathcal{H}_{12} - E\mathcal{N}_{12})\,(\mathcal{H}_{22} - E\mathcal{N}_{22})^{-1}\,(\mathcal{H}_{21} - E\mathcal{N}_{21})\,. \tag{3.9}$$

Equation (3.8) strongly differs from the Schrödinger equation because (1) it describes all possible states of the exciton gas including an electron–hole plasma and as a result operator $\mathcal{H}^{(XX)}$ is a nonlinear function of E, and because (2) it is written in a non-orthogonal basis and $\mathcal{N}_{11}$ is not the unity matrix. However, under conditions (3.2) and (3.3) the matrix elements of this operator are reduced to [20]

$$\mathcal{H}^{(XX)}_{\nu\nu'} = \sum_{\mu} \frac{(\mathcal{H}_{12} + \mathcal{N}_{12}NE_{\mathrm{b}})_{\nu\mu}(\mathcal{H}_{21} + \mathcal{N}_{21}NE_{\mathrm{b}})_{\mu\nu'}}{E_{\mu} + NE_{\mathrm{b}}}, \tag{3.10}$$

where ν and ν' enumerate states in subspace 1, μ and μ' enumerate states in subspace 2, and E_{μ} are eigenvalues of $\mathcal{H}_{22}$. That is, $\mathcal{H}^{(XX)}_{\nu\nu'}$ is the interaction Hamiltonian between ground-state excitons via excited states.

The normalization matrix $\mathcal{N}_{11}$ can be presented as

$$\mathcal{N}_{11} = I + \mathcal{A}, \tag{3.11}$$

where I is the unity matrix. The main contribution to the non-diagonal matrix $\mathcal{A} = \mathcal{A}^{\dagger}$ comes from the integrals of products of functions $\Phi_{\gamma_1,\ldots,\gamma_N}$, equation (2.4), that differ by transposition of only one pair of electrons or holes. The integration produces the factor a^2/S. Contributions to $\mathcal{A}$ containing integrals of products of the functions different by transposition of more than one pair of electrons or holes bring higher powers of a^2/S. Eventually, in the calculation of observable quantities, each factor of $1/S$ is accompanied by a sum over occupied states of the system, which gives the factor of N. Thus off-diagonal elements of $\mathcal{A}$ are characterized by the factor na^2. Terms of the same order can be separated in matrix $\mathcal{H}_{11}$:

$$\mathcal{H}_{11} = \mathcal{H}_0 + \mathcal{H}_1, \tag{3.12}$$

where $\mathcal{H}_0$ is the Hamiltonian of free excitons and $\mathcal{H}_1 \sim a^2/S$ describes their interaction. Neglecting terms of the second order in a^2/S, the diagonalization of equation (3.8) results in equation (3.4a), where the exciton Hamiltonian $\mathcal{H}_X$ contains a few contributions:

$$\mathcal{H}_X = \mathcal{H}_0 + \mathcal{H}_{\mathrm{int}} + \mathcal{H}_{\mathrm{ex}}. \tag{3.13}$$

Here $\mathcal{H}_0$ is the Hamiltonian of non-interacting excitons,

$$\mathcal{H}_{\mathrm{int}} = \mathcal{H}_1 + \mathcal{H}^{(XX)} \tag{3.14a}$$

is the Hamiltonian of potential interaction, and

$$\mathcal{H}_{\mathrm{ex}} = -\tfrac{1}{2}\left(\mathcal{A}\mathcal{H}_0 + \mathcal{H}_0\mathcal{A}\right) \tag{3.14b}$$

is the exchange interaction Hamiltonian. After the second quantization [20, 21],

$$\mathcal{H}_X = \sum_{k,s} E_k c^{\dagger}_{k,s} c_{k,s} + \frac{1}{2S} \sum_{\substack{k_1,k_2,q \\ s_1,s_2}} \left[U_d(q) - V_1 \right] c^{\dagger}_{k_2,s_2} c^{\dagger}_{k_1,s_1} c_{k_1-q,s_1} c_{k_2+q,s_2}$$

$$+ \frac{V_x}{4S} \sum_{k_1,k_2,q} \left[\sum_{s_1 s_2} c^{\dagger}_{k_1 s_1} c^{\dagger}_{k_2-s_1} c_{k_1-qs_2} c_{k_2+q-s_2} \right.$$

$$\left. - 4 \sum_{s} c^{\dagger}_{k_1 s} c^{\dagger}_{k_2-s} c_{k_1-qs} c_{k_2+q-s} + 2 \sum_{s_1 s_2} c^{\dagger}_{k_1 s_1} c^{\dagger}_{k_2 s_2} c_{k_1-qs_1} c_{k_2+qs_2} \right]. \tag{3.15}$$

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A natural question is about the effect of the non-bosonic nature of excitons on the exciton Hamiltonian. To answer this question, it is enough to notice that the difference between excitons and true bosons comes from the overlap integrals that make basis (2.4) non-orthogonal. These overlap integrals enter the matrix elements $\mathcal{H}_{\Gamma\Gamma'}$ in equation (2.11) and also make normalization matrix $\mathcal{N}$ different from the unit matrix that enters $\mathcal{H}_X$ via the exchange Hamiltonian $\mathcal{H}_{\mathrm{ex}}$. It is easy to see that, in the accepted approximation, the non-bosonic nature of excitons changes the values of the interaction matrix elements only.

It is important to note, however, that there exist overlap integrals of functions (2.4) different by transposition of two, three, or more electrons and/or holes, which are neglected in equation (3.15). In higher orders in na^2 these overlap integrals bring in the exciton Hamiltonian an interaction between three-, four-, and higher number of excitons, in spite of all this being provided by the Coulomb interaction, which has a two-particle nature.

Direct observation of excitons is carried out in optical experiments. Therefore it is very important to find out what is the effect of the non-bosonic exciton nature on the interaction between excitons and electromagnetic field. This problem is discussed in the next section.

4. Interaction of excitons with electromagnetic field

Interaction with an electromagnetic field leads to the recombination or creation of an exciton. In a dilute exciton gas the corresponding interaction Hamiltonian is (see, e.g., [22])

$$\mathcal{H}_{X-\mathrm{em}}^{(1)} = \sum_{\substack{k,s=\pm 1 \\ q,\kappa}} \left(M_{ks}^{q\kappa} a_{q\kappa}^{\dagger} c_{ks} + M_{ks}^{q\kappa *} c_{ks}^{\dagger} a_{q\kappa} \right), \tag{4.1}$$

where $a_{q\kappa}^{\dagger}$ and $a_{q\kappa}$ are creation and annihilation operators of a photon with wavevector q and polarization κ ($\kappa = \alpha$ for polarization vectors in the (q, z) plane and $\kappa = \beta$ for polarization vectors perpendicular to this plane), and $c_{ks}^{\dagger}$ and c_{ks} are creation and annihilation operators of a ground-state exciton with wavevector k and spin s. The orthogonalization of the exciton basis leads to a nonlinear exciton–photon Hamiltonian [9, 21, 23, 24],

$$\mathcal{H}_{X-\mathrm{em}} = \mathcal{H}_{X-\mathrm{em}}^{(1)} + \mathcal{H}_{X-\mathrm{em}}^{(2)}, \tag{4.2a}$$

$$\mathcal{H}_{X-\mathrm{em}}^{(2)} = \sum_{\substack{k_1\sigma_1;k_2\sigma_2 \\ k\sigma;q\kappa}} \left(M_{k_1\sigma_1;k_2\sigma_2}^{k\sigma;q\kappa} a_{q\kappa}^{\dagger} c_{k\sigma}^{\dagger} c_{k_1\sigma_1} c_{k_2\sigma_2} + M_{k_1\sigma_1;k_2\sigma_2}^{k\sigma;q\kappa *} c_{k_1\sigma_1}^{\dagger} c_{k_2\sigma_2}^{\dagger} a_{q\kappa} c_{k\sigma} \right). \tag{4.2b}$$

There are two clearly distinguished contributions to the Hamiltonian $\mathcal{H}_{X-\mathrm{em}}^{(2)}$ [21]. One comes from the exciton–exciton interaction and means that the amplitude of creation or recombination of an exciton depends on the presence of other excitons around it. The other contribution comes from the overlap (non-orthogonality) of the exciton wavefunctions and, in the case of recombination, means that an electron and a hole of two different excitons recombine, leaving behind one hole and one electron that form a new exciton.

The nonlinear exciton–photon interaction (4.2) leads to saturation of the exciton oscillator strength [9] and saturation of the exciton concentration under strong pumping [24]. This interaction can also substantially affect the collisional broadening of the exciton luminescence line. The point is that, in the linear recombination process described by the Hamiltonian (4.1), the momentum of the recombining exciton is strongly limited by the momentum conservation law, and this applies a very strong limitation on the photon energy. In the nonlinear recombination process the remaining exciton can carry any momentum and the limitation on the photon energy is lifted. Ciuti *et al* have shown that the linear collisional broadening produced a line with the width of the exciton spectral width, Γ [25]. The broadening coming from $\mathcal{H}_{X-\mathrm{em}}^{(2)}$ is controlled by the energy distribution of the remaining exciton, i.e. it can be of the order of

the exciton temperature. The expression for the luminescence intensity of the non-degenerate exciton gas in the direction perpendicular to the quantum well can be written as

$$I_{q\kappa} = S A^2 \omega_q^2 \lambda_\kappa \left[n_X \frac{\Gamma}{(\hbar\omega_q - E_{\mathrm{th}})^2 + \Gamma^2} + \frac{M a^2}{\hbar^2} \frac{n_X^2}{n_{X0}} r \left(\frac{|E_{\mathrm{th}} - \hbar\omega_q|}{T}, \frac{3\Gamma}{T} \right) \right], \tag{4.3}$$

where A^2 and $n_{X0} \approx \pi a^2$ are constants related to the matrix elements, M and a are the exciton mass and radius, $\lambda_\alpha = \cos^2\theta$, $\lambda_\beta = 1$, θ is the angle between the photon wavevector $\boldsymbol{q}$ and the normal to the quantum well, $\boldsymbol{q}_\parallel$ is the in-plane component of $\boldsymbol{q}$, $\hbar\omega_q$ is the photon energy, E_{th} is the threshold energy for exciton creation, n_X is the exciton concentration, and

$$r(u, v) = \int_0^\infty \mathrm{e}^{-x} \left[\frac{v}{(u+x)^2 + v^2} + \frac{v}{(u-x)^2 + v^2} \right] \mathrm{d}x. \tag{4.4}$$

The exciton spectrum is well defined when $\Gamma \ll T$, and in this case

$$r \left(\frac{|E_{\mathrm{th}} - \hbar\omega_q|}{T}, \frac{3\Gamma}{T} \right) = \mathrm{e}^{-|E_{\mathrm{th}} - \hbar\omega_q|/T}. \tag{4.5}$$

That is, except for the linear peak studied by Ciuti *et al* [25], the luminescence line contains a nonlinear background of the temperature width and the intensity that grows quadratically with exciton concentration.

5. Conclusion

In conclusion, the answer to the question posed in the title of the paper is the following: each separate exciton is not a boson, but the gas of excitons is a Bose gas. A formal difference between excitons and bosons comes from the non-orthogonality of multi-exciton wavefunctions constructed by symmetrization of the product of wavefunctions of separate excitons. The physics behind this is that, due to identity of electrons and holes in different excitons, true bosons are not separate excitons consisting of a given electron and a given hole but mixtures of all electrons and holes with the proper symmetry. In other words, the difference between separate excitons and true bosons comes from exchange interaction between their constituents. Formally, the multi-boson wavefunctions are obtained by orthogonalization of the multi-exciton wavefunctions. Additional terms in the Hamiltonian resulting from the orthogonalization are just corrections to exciton–exciton interaction matrix elements. The exciton–exciton interaction and the non-bosonic nature of separate excitons lead also to a nonlinear interaction between bosons and photons. Among the effects of this interaction, there is a collisionally broadened luminescence with an intensity proportional to the exciton concentration squared and the width of the order of the exciton temperature.

Appendix. Two-exciton states basis

The completeness of the basis equation (2.2) is clear from the following consideration. Functions $\psi_\gamma(e, h)$ comprise a complete basis in the space of all states of one electron–hole pair. That is, any two-exciton wavefunction $f(e_1, h_1; e_2, h_2)$ can be represented as a series in $\psi_{\gamma_1}(e_1, h_1)$ with coefficients depending on e_2, h_2. For the same reason, each of these coefficients can be represented as a series in $\psi_{\gamma_2}(e_2, h_2)$. Due to the antisymmetry of $f(e_1, h_1; e_2, h_2)$ with respect to the transposition of electrons and the transposition of holes, such an expansion is reduced to an expansion in functions (2.2):

$$f(e_1, h_1; e_2, h_2) = \sum_{(\gamma_1\gamma_2)} f_{(\gamma_1\gamma_2)} \Phi_{(\gamma_1,\gamma_2)}(e_1, h_1; e_2, h_2). \tag{A.1}$$

The basis functions are symmetric with respect to transposition quantum numbers γ_1 and γ_2; the pair of them can be considered as a composite quantum number of a two-exciton state. To emphasize this point, the pair of single exciton quantum numbers forming a two-exciton quantum number is put in parentheses in this appendix. The summation $\sum_{(\gamma_1\gamma_2)}$ implies the summation over all different pairs of γ_1 and γ_2, i.e.

$$\sum_{(\gamma_1,\gamma_2)} = \sum_{\gamma_1=\gamma_2} + \tfrac{1}{2} \sum_{\gamma_1 \neq \gamma_2} . \tag{A.2}$$

The basis functions are not orthogonal,

$$\int \Phi^*_{(\gamma_1,\gamma_2)}(e_1, h_1; e_2, h_2) \Phi_{(\gamma_3,\gamma_4)}(e_1, h_1; e_2, h_2)\, \mathrm{d}e_1\, \mathrm{d}h_1\, \mathrm{d}e_2\, \mathrm{d}h_2 = \mathcal{N}_{(\gamma_1,\gamma_2),(\gamma_3,\gamma_4)}, \tag{A.3}$$

and to find the the coefficients in expansion (A.1) it is necessary to introduce a dual basis,

$$\tilde{\Phi}_{(\gamma_1,\gamma_2)}(e_1, h_1; e_2, h_2) = \sum_{(\gamma_3,\gamma_4)} \mathcal{N}^{-1}_{(\gamma_1,\gamma_2),(\gamma_3,\gamma_4)} \Phi^*_{(\gamma_3,\gamma_4)}(e_1, h_1; e_2, h_2) \tag{A.4}$$

that satisfies the orthogonality and normalization relation:

$$\int \tilde{\Phi}_{(\gamma_1,\gamma_2)}(e_1, h_1; e_2, h_2) \Phi_{(\gamma_3,\gamma_4)}(e_1, h_1; e_2, h_2)\, \mathrm{d}e_1\, \mathrm{d}h_1\, \mathrm{d}e_2\, \mathrm{d}h_2 = \delta_{(\gamma_1,\gamma_2),(\gamma_3,\gamma_4)}. \tag{A.5}$$

The unit matrix in the symmetric space is expressed in regular Kronecker symbols as

$$\delta_{(\gamma_1,\gamma_2),(\gamma_3,\gamma_4)} = \frac{\delta_{\gamma_1\gamma_3}\delta_{\gamma_2\gamma_4} + \delta_{\gamma_1\gamma_4}\delta_{\gamma_2\gamma_3}}{\sqrt{(1+\delta_{\gamma_1\gamma_2})(1+\delta_{\gamma_3\gamma_4})}}. \tag{A.6}$$

Then

$$f_{(\gamma_1,\gamma_2)} = \int \tilde{\Phi}_{(\gamma_1,\gamma_2)}(e_1, h_1; e_2, h_2) f(e_1, h_1; e_2, h_2)\, \mathrm{d}e_1\, \mathrm{d}h_1\, \mathrm{d}e_2\, \mathrm{d}h_2. \tag{A.7}$$

Substitution of equation (A.7) in equation (A.1) gives

$$f(e_1, h_1; e_2, h_2) = \int f(e_3, h_3; e_4, h_4)\, \mathrm{d}e_3\, \mathrm{d}h_3\, \mathrm{d}e_4\, \mathrm{d}h_4$$
$$\times \sum_{(\gamma_1\gamma_2)} \Phi_{(\gamma_1,\gamma_2)}(e_1, h_1; e_2, h_2) \tilde{\Phi}_{(\gamma_1,\gamma_2)}(e_3, h_3; e_4, h_4). \tag{A.8}$$

This relation takes place for any function $f(e_1, h_1; e_2, h_2)$ that has the necessary symmetry. Therefore the sum equals the symmetrized δ-function,

$$\sum_{(\gamma_1\gamma_2)} \Phi_{(\gamma_1,\gamma_2)}(e_1, h_1; e_2, h_2) \tilde{\Phi}_{(\gamma_1,\gamma_2)}(e_3, h_3; e_4, h_4)$$
$$= \tfrac{1}{4}\Big[\delta(e_1 - e_3)\delta(e_2 - e_4)\delta(h_1 - h_3)\delta(h_2 - h_4)$$
$$- \delta(e_1 - e_4)\delta(e_2 - e_3)\delta(h_1 - h_3)\delta(h_2 - h_4)$$
$$- \delta(e_1 - e_3)\delta(e_2 - e_4)\delta(h_1 - h_4)\delta(h_2 - h_3)$$
$$+ \delta(e_1 - e_4)\delta(e_2 - e_3)\delta(h_1 - h_4)\delta(h_2 - h_3)\Big], \tag{A.9}$$

i.e. the δ-function in symmetric space. Equation (A.9) is the completeness relation of the $\Phi_{(\gamma_3,\gamma_4)}(e_1, h_1; e_2, h_2)$ basis.

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